

A REMARK ON APPROXIMATION ON TOTALLY REAL SETS.

BO BERNDTSSON

ABSTRACT. We give a new proof of a theorem on approximation of continuous functions on totally real sets.

1. INTRODUCTION

Let Ω be a pseudoconvex open set in \mathbb{C}^n , and let ϕ be a C^2 -smooth non-negative plurisubharmonic function in Ω , satisfying $i\partial\bar{\partial}\phi \geq \delta\beta$, where β is the Euclidean volume form and $\delta > 0$. Then

$$E := \{\phi = 0\}$$

is a *totally real* set.

Associated to the function ϕ and to any positive number k we have the orthogonal projection operator P_k from

$$L^2(\Omega, e^{-k\phi})$$

to

$$A^2(\Omega, e^{-k\phi}),$$

the latter space being the Bergman space, i.e. the subspace of holomorphic functions in $L^2(\Omega, e^{-k\phi})$.

We shall prove the following theorem:

Theorem 1.1. *Let u be a smooth function of compact support in Ω . Let K be a compact subset of Ω . Then*

$$\sup_{E \cap K} |u - P_k(u)| \leq C/\sqrt{k},$$

for some constant C .

In particular it follows that any continuous function on E can be approximated uniformly on compacts of E by functions holomorphic in Ω . Since any totally real submanifold of class C^1 can be given as the zeroset of a strictly plurisubharmonic function defined in some neighbourhood of the manifold, Theorem 1.1 contains the theorem of Hörmander and Wermer, [7] and Nirenberg-Wells, [9], as well as the generalization to the C^1 -case of Harvey and Wells, [6].

Just like in the original proofs in [7] and [9], the proof of Theorem 1.1 is based on Hörmander's L^2 -estimates for the $\bar{\partial}$ -operator. One difference between the proofs is that we will use the weight factor $e^{-k\phi}$ in the estimates. We shall then use the following consequence of Hörmander's theorem.

Theorem 1.2. *Let ϕ be a plurisubharmonic function in a pseudoconvex domain Ω , satisfying $i\partial\bar{\partial}\phi \geq \delta\beta$. Let f be a $\bar{\partial}$ -closed form of bidegree $(0, 1)$ in Ω , and let v be the L^2 -minimal solution to the equation $\bar{\partial}v = f$ in $L^2(\Omega, e^{-k\phi})$. Then, for $k > 0$,*

$$\int |v|^2 e^{-k\phi} \leq C/k \int |f|^2 e^{-k\phi}.$$

The important feature of this theorem here is that the estimates gets better as k increases. The same estimate holds if we replace $k\phi$ by ϕ_k where $i\partial\bar{\partial}\phi_k \geq k\delta\beta$.

The next step in the proof is the observation that something similar happens in uniform norms, at least if we shrink the domain a little. This is not entirely trivial, but nor is it a deep observation - the shrinking of the domain avoids the main difficulty in passing from L^2 to uniform norms. The main point in the proof is a variant of the Donnelly-Fefferman trick.

Theorem 1.3. *Let ϕ be a plurisubharmonic function in a pseudoconvex domain Ω , satisfying $i\partial\bar{\partial}\phi \geq \delta\beta$. Let v be the $L^2(\Omega, e^{-k\phi})$ -minimal solution to the equation $\bar{\partial}v = f$. Then, if K is a compact subset of Ω*

$$\sup_K |v|^2 e^{-k\phi} \leq \frac{C_{\delta,K}}{k} \sup_{\Omega} |f|^2 e^{-k\phi}.$$

We apply Theorem 1.3 to $f = \bar{\partial}u$ where u is, say, a test function. Since

$$u - P_k(u) = v,$$

it follows that with $E = \phi^{-1}(0)$

$$\sup_{E \cap K} |u - P_k(u)|^2 \leq \frac{C_{\delta,K}}{k} \sup_{\Omega} |f|^2 e^{-k\phi} \leq \frac{C_{\delta,K}}{k} \sup_{\Omega} |f|^2,$$

if $\phi \geq 0$, and Theorem 1.1 follows.

It is interesting to note that the proof of the approximation theorem here also has some features in common with the proof of (a generalization of) the Hörmander-Wermer theorem of Baouendi and Treves, [1]. Their proof is based on convolution with a Gaussian kernel, whereas here we apply the Bergman kernel. However, in the model case $\phi = x^2$, the Bergman kernel is Gaussian, so the two proofs are actually quite similar in this case.

If K is a compact subset of E , which is moreover polynomially convex, we can choose the weight function ϕ so that it has logarithmic growth at infinity. The holomorphic functions $P_k(u)$ are then polynomials of degree k , and Theorem 1.1 estimates the degree of approximation by $1/\sqrt{k}$, if u is of class C^1 . This is not quite as good as one would expect; at least if E is a smooth manifold the right degree of approximation should be $1/k$. Possibly this flaw comes from letting E be a quite general set. At any rate it seems hard to do better with the methods in this paper.

One might also notice that Theorem 1.3 works equally well in unbounded domains, so minor modifications should give Carleman-Type approximation as well, see [4], [8].

Finally, I would like to thank Said Asserda and the referee for pointing out several inaccuracies and obscurities in the first version of this paper.

2. PROOF OF THEOREM 1.3

Theorem 1.3 is not really new - it follows readily from results in [2] and [3] and in particular [5] - but here we shall indicate a concise proof for the case at hand. The main ingredient is a variant of the Donnelly-Fefferman trick which will give us an Agmon-type estimate.

Theorem 2.1. *Assume $i\partial\bar{\partial}\phi \geq 5\beta$. Then, with notation as in the introduction, for any a in \mathbb{C}^n*

$$(2.1) \quad \int |v|^2 e^{-k\phi - \sqrt{k}|z-a|} \leq C/k \int |f|^2 e^{-k\phi - \sqrt{k}|z-a|}.$$

Proof. Assume for simplicity of notation that $a = 0$. Let ψ be the convex function defined by

$$\psi(t) = t$$

for $t \geq 1$ and

$$\psi(t) = t^2/2 + 1/2$$

for $t < 1$. Let $\chi(z) = \psi(|z|)$. Then $|\chi - |z||$ is bounded, so it is enough to prove 2.1 with $\sqrt{k}|z|$ replaced by $\chi(z\sqrt{k})$. Moreover $\partial\chi$ is bounded by 1, and $i\partial\bar{\partial}\chi \leq \beta$. It is especially the last property that is of importance here and is the reason for introducing the function χ . Put

$$\chi_k(z) = \chi(z\sqrt{k})$$

and

$$v_k := v e^{-\chi_k}.$$

Since v is orthogonal to all holomorphic functions in $L^2(\Omega, e^{-k\phi})$, it follows that v_k is orthogonal to all holomorphic functions for the scalar product in $L^2(\Omega, e^{-k\phi + \chi_k})$ (we may assume in the proof that Ω is bounded so that the L^2 -spaces do not change when we vary the weight). Hence v_k is the L^2 minimal solution to a certain $\bar{\partial}$ -equation. Now,

$$i\partial\bar{\partial}(k\phi - \chi_k) \geq Ck\beta$$

so it follows from the Hörmander estimate 1.2 that

$$(2.2) \quad \int |v_k|^2 e^{-k\phi + \chi_k} \leq C/k \int |\bar{\partial}v_k|^2 e^{-k\phi + \chi_k}.$$

The left hand side of equation 2.2 equals

$$\int |v|^2 e^{-k\phi - \chi_k}.$$

In the right hand side we have

$$\bar{\partial}v_k = (f - v_k \bar{\partial}\chi_k) e^{-\chi_k}.$$

Since $\bar{\partial}\chi_k$ is bounded by \sqrt{k} we can absorb the contribution to 2.2 coming from the second term $v_k \bar{\partial}\chi_k$ in the left hand side of 2.2, and 2.1 follows.

□

We are now ready for the proof of Theorem 1.3. By scaling, we may of course assume that $i\partial\bar{\partial}\phi \geq 5\beta$. Let a be a point in Ω_k that again for simplicity we take equal to 0. After changing frame locally at 0 we can assume that, near the origin,

$$\phi(z) = q(z, \bar{z}) + o(|z|^2),$$

where q is an hermitian form.

(This means the following: Near the origin we can, since ϕ is of class C^2 , write

$$\phi(z) = 2\operatorname{Re} P(z) + q(z, \bar{z}) + o(|z|^2),$$

where P is a holomorphic polynomial of degree 2. We then write

$$v' = ve^{-P(z)}$$

for v and

$$f' = fe^{-P(z)}$$

for f , which changes the weight function ϕ to $\phi - 2\operatorname{Re} P = q + o(|z|^2)$.)

Note that $kq(z, \bar{z})$ is bounded by a constant when $|z| < 1/k$. This constant certainly depends on the point a that we have taken equal to 0, but it is uniform as long as a ranges over a compact subset of Ω . We then get

$$\int_{|z|^2 < 1/k} |v|^2 \leq C/k \sup |f|^2 e^{-k\phi} \int e^{-\sqrt{k}|z|} \leq C \sup |f|^2 e^{-k\phi} / k^{n+1}.$$

Normalize so that

$$\sup |f|^2 e^{-k\phi} \leq 1.$$

Then, in particular, $\bar{\partial}v$ is bounded by a constant for $|z|^2 < 1/k$. The inequality

$$(2.3) \quad |v(0)|^2 \leq C(k^n \int_{|z|^2 < 1/k} |v|^2 + \frac{1}{k} \sup_{|z|^2 < 1/k} |f|^2),$$

then shows that

$$|v(0)|^2 \leq 1/k,$$

which is what we wanted to prove. To verify 2.3 one can apply the Bochner-Martinelli integral formula

$$v(0) = c_n \int (\bar{\partial}(v\xi(|z|^2/k)) \cdot \partial|z|^{2n-2}),$$

where $\xi(t)$ is a smooth function that equals 1 for $t < 1/2$ and 0 for $t > 1$.

REFERENCES

- [1] BAOUENDI, M. S.; TRÈVES, F.: *A property of the functions and distributions annihilated by a locally integrable system of complex vector fields*, Ann. of Math. (2) 113 (1981), no. 2, 387–421.
- [2] BERNDTSSON, BO: *Weighted estimates for the $\bar{\partial}$ -equation*, Complex analysis and geometry (Columbus, OH, 1999), 43–57, Ohio State Univ. Math. Res. Inst. Publ., 9, de Gruyter, Berlin, 2001.
- [3] BERNDTSSON, BO: *Uniform estimates with weights for the $\bar{\partial}$ -equation*, J. Geom. Anal. 7 (1997), no. 2, 195–215.
- [4] CARLEMAN, T: *Sur un Theoreme de Weierstrass*, Arkiv för Matematik, Astronomi och Fysik 20B, 4 (1927) .
- [5] DELIN, HENRIK: *Pointwise estimates for the weighted Bergman projection kernel in \mathbb{C}^n , using a weighted L^2 estimate for the $\bar{\partial}$ equation*, Ann. Inst. Fourier (Grenoble) 48 (1998), no. 4, 967–997.
- [6] HARVEY, F. REESE; WELLS, R. O., JR.: *Holomorphic approximation and hyperfunction theory on a C^1 totally real submanifold of a complex manifold*, Math. Ann. 197 (1972), 287–318.
- [7] HÖRMANDER, L.; WERMER, J.: *Uniform approximation on compact sets in \mathbb{C}^n* , Math. Scand. 23 1968 5–21 (1969) .
- [8] MANNE, P: *Carleman approximation on totally real submanifolds of a complex manifold*, Several complex variables (Stockholm, 1987/1988), 519–528, Math. Notes, 38, Princeton Univ. Press, Princeton, NJ, 1993 .
- [9] NIRENBERG, RICARDO; WELLS, R. O., JR.: *Holomorphic approximation on real submanifolds of a complex manifold*, Bull. Amer. Math. Soc. 73 1967 378–381.

B BERNDTSSON :DEPARTMENT OF MATHEMATICS, CHALMERS UNIVERSITY OF TECHNOLOGY AND THE UNIVERSITY OF GÖTEBORG, S-412 96 GÖTEBORG, SWEDEN,,

E-mail address: bob@math.chalmers.se